MATH 2850: CHAPTER 4: APPLICATIONS

GROWTH MODELS: Recall from Chapter 2:

- **UNLIMITED GROWTH:**

  BASIC PRINCIPLE: rate of change is directly proportional to amount present:

  initial amount $y_0 > 0$, growth rate: k:

$$\frac{dy}{dt} = k\,y,\ y(0) = y_0 \implies y = y_0\,e^{kt}$$

  Classic examples: continuously compounded interest $A = Pe^{rt}$, radioactive decay $A = A_0 e^{-kt}$

- **LIMITED GROWTH:**

  BASIC PRINCIPLE: rate of change is directly proportional to room left to grow:

  initial amount: y_0, $0 < y_0 < L$, limiting amount L, rate of proportionality k:

$$\frac{dy}{dt} = k\,(L - y)\,,\ y(0) = y_0 \implies y = L + (y_0 - L)\,e^{-kt}$$

  Classic example: Newton's Law of Cooling: $T(t) = T_a + (T_0 - T_a)e^{-kt}$

- **LOGISTIC GROWTH:**

  BASIC PRINCIPLE: rate of change is jointly proportional to amount present **and** room left to grow:

  initial amount: y_0, $0 < y_0 < L$, limiting amount L, rate of proportionality k:

$$\frac{dy}{dt} = k\,y\,(L - y),\ y(0) = y_0 \implies y = \frac{C\,L\,e^{kLt}}{1 + C\,e^{kLt}},\ C = \frac{y_0}{L - y_0} = \text{ratio of what we start with to room to grow}$$

MIXING PROBLEMS: Basic principle: NET RATE = RATE IN − RATE OUT

EXAMPLE: A 100 gallon tank is initially full of a brine (salt) solution in which 25 pounds of salt are dissolved.

Suppose an inlet valve is opened which delivers $\frac{1}{2}$ pound of salt per gallon at a rate of 1 gallon per minute.

Suppose an outlet valve is opened which drains the well-mixed solution also at a rate of 1 gallon per minute.

Write and solve a differential equation to solve for $A(t)$, the amount of salt in the tank at time t.

What happens as $t \to \infty$?

Ans: $\dfrac{dA}{dt} = \dfrac{1}{2} - \dfrac{A}{100}$, $A(0) = 25 \implies A(t) = 50 - 25e^{-t/100}$.

As $t \to \infty$, $A(t) \to 50$ so the tank will 'eventually' contain 50 lbs of salt.

MIXING PROBLEM (REPRISE): Basic principle: NET RATE = RATE IN − RATE OUT

EXAMPLE: A 100 gallon tank is initially full of a brine (salt) solution in which 25 pounds of salt are dissolved.

Suppose an inlet valve is opened which delivers $\frac{1}{2}$ pound of salt per gallon at a rate of 1 gallon per minute.

Suppose an outlet valve is opened which drains the well-mixed solution at a rate of $\underline{2}$ gallons per minute.

Write and solve a differential equation to solve for $A(t)$, the amount of salt in the tank at time t.

Solve $A(t) = 0$ and interpret.

Ans: $\dfrac{dA}{dt} = \dfrac{1}{2} - \dfrac{2A}{100-t}$, $A(0) = 25 \implies A(t) = 25 - \dfrac{1}{400}t^2$.

$A(t) = 0$ when $t = 100$ which means there's no longer salt in the tank.

We're losing 1 gallon per minute so after 100 minutes, we've drained 100 gallons - that is, the whole tank.

ONE DIMENSIONAL (VERTICAL) MOTION:

NEWTON'S SECOND LAW OF MOTION:

If an object has mass m, velocity, v, and acceleration a, the force acting on the object F satisfies: $F = D_t\,[mv]$.

Assuming a constant mass, this reduces to: $F = D_t\,[mv] = m\,D_t[v] = ma$.

Let $y(t)$ denote the height of an object off the ground at time t. Then $v(t) = y'(t)$ and $a(t) = v'(t) = y''(t)$.

Assuming constant mass, no wind resistance and constant gravity, we apply Newton's Second Law:

$$F = ma \iff -mg = my'' \iff y'' = -g$$

Imposing initial conditions: $y'(0) = v_0$ and $y(0) = h$, we get $y(t) = -\frac{1}{2}gt^2 + v_0 t + h$.

WIND RESISTANCE AND TERMINAL VELOCITY:

Suppose we add in a term to account for wind resistance. There are two scenarios:

- the wind resistance force is directly proportional to the velocity (low drag): $F = kv$
- the wind resistance force is directly proportional to the velocity squared (high drag) : $F = kv^2$

We'll focus on the second scenario. Here, the constant k must have units that convert from (velocity)^2 units to units of force. Breaking things down, we get:

$$k = \frac{1}{v^2}\,F \sim \frac{(\text{time})^2}{(\text{length})^2}\,\frac{(\text{mass})\,(\text{length})}{(\text{time})^2} \sim \frac{\text{mass}}{\text{length}}$$

Since wind resistance would **oppose** gravity, and remembering $a(t) = v'(t)$, Newton's Second Law becomes:

$$F = ma \iff -mg + kv^2 = ma \iff a = -g + \frac{k}{m}v^2 \iff v' = -g\left[1 - \frac{k}{mg}v^2\right] \iff v' = -g\left[1 - \alpha^2 v^2\right]$$

where we have relabeled $\frac{k}{mg} = \alpha^2$ for convenience. Checking units, we get:

$$\alpha = \sqrt{\frac{k}{mg}} \sim \sqrt{\frac{\text{mass}}{\text{length}}\,\frac{1}{\text{mass}}\,\frac{(\text{time})^2}{\text{length}}} \sim \sqrt{\frac{(\text{time})^2}{(\text{length})^2}} \sim \frac{\text{time}}{\text{length}}$$

We couple this DE with the IC $v(0) = 0$ (that is, we are dropping an object off of a cliff, or out or an airplane).

$$v' = -g\left[1 - \alpha^2 v^2\right],\ v(0) = 0$$

To find the **terminal velocity** of a falling object, v_∞, we take the limiting value of our solution, $v(t)$: $\lim_{t\to\infty} v(t)$.

EXAMPLE: Solve the IVP: $v' = -g\left[1-\alpha^2 v^2\right]$, $v(0)=0$ and find $v_\infty = \lim_{t\to\infty} v(t)$.

Ans: $v(t) = \dfrac{1}{\alpha}\left[\dfrac{e^{-2\alpha g t}-1}{e^{-2\alpha g t}+1}\right]$, $v_\infty = \lim_{t\to\infty} v(t) = -\dfrac{1}{\alpha}$.

Substituting in for α, we get $v_\infty = -\dfrac{1}{\alpha} = -\sqrt{\dfrac{mg}{k}}$.

To make more sense of this, we write $k = \dfrac{1}{2}\,\rho\, A\, C_d$.

Here, ρ is the density of the air, A is the 'projected area' of the object, and C_d is the drag coefficient.

EXAMPLE: Substitute $k = \dfrac{1}{2}\,\rho\, A\, C_d$ into the formula for $v_\infty = -\sqrt{\dfrac{mg}{k}}$ and simplify a formula for v_∞.

Ans: $v_\infty = -\sqrt{\dfrac{2gm}{\rho A C_d}}$

NOTE: You can also get terminal velocity **algebraically** by equating gravitational force and the buoyant force.

ESCAPE VELOCITY:

Now suppose we wish launch an object with constant mass m off into space. Let $y(t)$ represent the distance of the object off of the ground with $y(0) = y_0$ and let $v(0) = v_0$ be the initial velocity of the object.

In this scenario, we no longer assume the force of gravity is constant. Indeed, the force of gravity is given by:

$$F = -\frac{GmM}{(y+R)^2}$$

where G is the gravitational constant, M is the mass of the Earth, and R is the radius of the Earth. We get:

$$my'' = -\frac{GmM}{(y+R)^2} \iff y'' = -\frac{GM}{(y+R)^2}$$

If we let $v = y' = \dfrac{dy}{dt}$, then $y'' = \dfrac{d^2y}{dt^2} = \dfrac{dv}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt} = \dfrac{dv}{dy}v = v\dfrac{dv}{dy}$. Hence, our DE becomes

$$v\frac{dv}{dy} = -\frac{GM}{(y+R)^2}$$

EXAMPLE: Find an implicit solution to: $v\dfrac{dv}{dy} = -\dfrac{GM}{(y+R)^2}$ with $y(0) = y_0$ and let $v(0) = v_0$.

Ans: $\dfrac{1}{2}v^2 = \dfrac{GM}{y+R} + \left[\dfrac{1}{2}v_0^2 - \dfrac{GM}{y_0+R}\right]$

EXAMPLE: Using the solution $\dfrac{1}{2}v^2 = \dfrac{GM}{y+R} + \left[\dfrac{1}{2}v_0^2 - \dfrac{GM}{y_0+R}\right]$, show that:

- if $v_0 \geq \sqrt{\dfrac{2GM}{y_0+R}}$, then $v(t) > 0$ always which means the object never returns to the surface of the Earth.

- if $v_0 < \sqrt{\dfrac{2GM}{y_0+R}}$, then $v(t) = 0$ at some point so the object returns to the surface of the Earth.

Hence, $v_0 = \sqrt{\dfrac{2GM}{y_0+R}}$ is the so-called **escape** velocity.

NOTE: Rewriting our solution, we get:

$$\frac{1}{2}v^2 = \frac{GM}{y+R} + \left[\frac{1}{2}v_0^2 - \frac{GM}{y_0+R}\right] \iff \frac{1}{2}v^2 - \frac{1}{2}v_0^2 = -\left[\frac{GM}{y_0+R} - \frac{GM}{y+R}\right]$$

Multiplying through my m, we get:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -\left[\frac{GMm}{y_0+R} - \frac{GMm}{y+R}\right]$$

What we have here is a statement about the conservation of energy!

difference in kinetic energy $= \dfrac{1}{2}mv^2 - \dfrac{1}{2}mv_0^2 = -\left[\dfrac{GMm}{y_0+R} - \dfrac{GMm}{y+R}\right] =$ difference in gravitational potential energy

ORTHOGONAL TRAJECTORIES:

RECALL: If a line L has slope $m \neq 0$, then the slope of the line perpendicular to L has slope $-\dfrac{1}{m}$.

RECALL: If the curve $F(x, y) = C$ implicitly describes y as a function of x, then $\dfrac{dy}{dx} = -\dfrac{F_x(x, y)}{F_y(x, y)}$.

Consider the family of circles $x^2 + y^2 = C$.

Identifying $F(x, y) = x^2 + y^2$, We may describe the slopes of the tangent lines of this family by the DE:

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{x}{y}$$

Suppose we wish to find a family of curves which intersect these circles at $90°$? To do this, we solve:

$$\frac{dy}{dx} = -\frac{1}{-x/y} = \frac{y}{x}$$

EXAMPLE: Solve $\dfrac{dy}{dx} = \dfrac{y}{x}$.

Graph some representative integral curves along with circles of the form $x^2 + y^2 = C$.

Observe both curves near the points of intersection.

IN GENERAL:

If a curve is described by $\dfrac{dy}{dx} = f(x, y)$, the family of **orthogonal trajectories** are described by: $\dfrac{dy}{dx} = -\dfrac{1}{f(x, y)}$.

HOMEWORK: Pg. 138: 11; pg. 148: 5, 9, 11; pg. 152: read Example 4.3.1, pg. 169: 8, 9; pg. 192: 25, 29